Formal Analysis
Microeconomics: Walrasian Equilibrium and Market Failures

Ben Ansell

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1 Introduction

In this lecture, I provide a quick overview of the classic microeconomic depiction of the market and market failures. I follow Przeworski's Chapter Two as well as the book he cites by Per-Olov Johansson and Chapter Two from Mueller. In general the key concepts you need to be able to follow are the effects of prices and income on consumption decisions; how the market enables prices to match citizens' marginal rates of substitution across goods (the first fundamental theorem of welfare economics), and how increasing returns, public goods, and externalities, alter the consumer's optimal choice. These latter issues provide the bedrock rationalization for the existence of government in the standard welfare economics and public finance view.

2 Models of Consumer Behavior

2.1 Maximization and the Marginal Rate of Substitution

We left the last class talking about utility functions over single goods. In the classic microeconomic model of consumer choice, utility functions are defined as $\mathbf{R}^n \rightarrow \mathbf{R}^1$ functions that convert the choice of n goods into a single scale. For the most part, economists simplify and use two goods, from which consumers choose quantities x_1 and x_2. A utility function is thus $U(x_1, x_2)$. Utility functions generally allow consumers to choose different bundles of goods that produce the same utility: that is, $U(x_1, x_2) = U(x_1', x_2')$. This gives us the concept of the indifference curve, which is a function in $\{x_1, x_2\}$ space where $U(x_1, x_2) = \bar{U}$, or some fixed level of utility.

The standard rules of utility functions from the previous class apply with the addition that we generally assume that utility functions are *convex to the origin*, meaning that they bend inward. This means that individual would prefer a mix of goods to one good outright and that the 'cost' of giving up one good for the other becomes lower, the more they have of the first good. This gives us the concept of a diminishing *marginal rate of substitution* or MRS. We calculate the MRS as the derivative of x_2 with respect to x_1 keeping utility constant- that is, how much x_2 would you be willing to give up for an extra unit of x_1?

Przeworski shows how you calculate this using substitution - I prefer to use what's called the implicit function theorem as it's more generalizable and actually easier to calculate. Put simply the implicit function theorem says that when you have an expression like $U(x, y) = k$, you can calculate the effect of x on y, holding U constant as $\frac{dy}{dx} = -\frac{\partial U/\partial x}{\partial U/\partial y}$. We can simply apply this idea to the basic consumer choice problem of moving along an indifference curve. We are taking the implicit function of $\bar{U}(x_1, x_2)$, which is constant. This provides us with the following fundamental result.

$$MRS = \left.\frac{dy}{dx}\right|_{U=\bar{U}} = -\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} \tag{1}$$

It's worth thinking about what this means - a consumer would be willing to give up more units of y to get one of x when the marginal utility they obtain from x is particularly high compared to that they get of y. If the utility function is set up to produce diminishing marginal utility form any one good holding the other constant- like the Cobb-Douglas form - then this implies that the MRS will also be decreasing as the quantity of that good being currently consumed rises. Symmetrically, a decreasing MRS implies diminishing marginal returns to any one good. We will see mathematically how to solve the consumer's problem under budget constraints shortly - but already you can note that the slope of the indifference curve, because it reflects the relative benefits derived from each good, will likely have to, in equilibrium, equal the marginal costs of acquiring each good- a.k.a. their prices. If the price of a good were less than the marginal utility of consuming it, holding the other good's price constant - then a consumer should consume more of that good until the MRS ratio equals the price ratio. We'll come back to this key condition.

2.2 Constrained Optimization and the Consumer's Choice

With this set-up established we can run through a basic consumer utility maximization problem, which I borrow from the appendix in Johansson. We continue with our basic two-good model, and now define utility as Cobb Douglas $U = x_1^{\alpha} x_2^{(1-\alpha)}$. We also introduce an income of y and prices for each good of p_1 and p_2. Consequently, consumers face the budget constraint of $y = p_1 x_1 + p_2 x_2$. To solve for this in the classic fashion we use the Lagrangian technique (you can also do this through substitution but that is quite painful, though that's what Przeworksi does!). We write out the Lagrangian as:

$$L = x_1^{\alpha} x_2^{(1-\alpha)} + \lambda(y - p_1 x_1 - p_2 x_2) \tag{2}$$

The way we solve a Lagrangian is to take its derivatives with respect to the choice variables x_1, x_2 and also with respect to the constraint parameter λ.

$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{(1-\alpha)} - p_1 \lambda = 0 \tag{3}$$

$$\frac{\partial L}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{(-\alpha)} - p_2 \lambda = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda} = y - p_1 x_1 - p_2 x_2 = 0 \tag{5}$$

Now we perform a little trick with the first two equations. We move the $-\lambda p_i$ elements over to the right hand side of the equation, and then because we have two equalities we can divide one by the other and still have an equality. Consequently, we divide the first equation by the second and we get the following result:

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2} \tag{6}$$

This produces an equation without y or λ showing precisely how relative prices and relative demands for goods are related. To interpret this equation let's return to the concept of the Marginal Rate of Substitution that we examined earlier (the slope of the indifference curve). We noted that you can solve for the MRS in the following manner.

$$MRS = -\left.\frac{dx_2}{dx_1}\right|_{U=\bar{U}} = \frac{\partial U / \partial x_1}{\partial U / \partial x_2} \tag{7}$$

So let us calculate the MRS using the original utility function: $\partial U / \partial x_1 = \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \alpha U / x_1$ and $\partial U / \partial x_1 = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = (1-\alpha) U / x_2$. Thus:

$$\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = MRS \tag{8}$$

But now look above to the optimality equation produced by the Lagrangian and we find that

$$MRS = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1} = \frac{p_1}{p_2} \tag{9}$$

This is the core result of consumer theory from which most of general equilibrium and welfare economics follows. Consumer behavior will be maximized when demand choices are made so that the ratio of prices equals the ratio of marginal utilities. Let us assume that the price of x_1, p_1 was less than the marginal utility of consuming another unit of x_1, $\partial U/\partial x_1$. Well then it would make sense to consume relatively more x_1 until the marginal utility of doing so had declined to the point where it equalled p_1. This is an equilibrium concept - individuals will adjust their consumption choices until they equalize their private relative benefit of consuming goods to the universal relative prices established for those goods.

2.3 Substitution and Income Effects

NOTE: ADVANCED (CAVEAT EMPTOR!!)

One further extension of this model is worth noting and that is to focus on how the demand for any given good changes as its price changes. Holding p_2 constant and varying p_1 we will see changes in the optimal choice of x_1. How do we explain these changes? It turns out that we can split a change in demand for a given good into two effects: (a) the *substitution effect* and (b) the *income effect*. If p_1 goes down, the optimal choice of x_1 will increase partly as consumers by less of x_2 and more of x_1 due to their changes in relative prices (the substitution effect) and partly because with a fixed income but cheaper x_1, consumers can afford more 'stuff' in general (the income effect).

We begin by solving for x_1. From the $MRS = p_1/p_2$ equation above we get $x_1 = \frac{\alpha}{1-\alpha}\frac{p_2}{p_1}x_2$, which is not desperately helpful. However, by moving things around we can get x_1 expressed as a function of y, α and p_1 (note this is just because we are using a Cobb-Douglas function - it may not be as simple more generally). We do the following algebra: $p_1 x_1 \frac{1-\alpha}{\alpha} = p_2 x_2$, then $p_1 x_1 \frac{1}{\alpha} = p_1 x_1 +$

$p_2x_2 = y$, which gives us our final result of $x_1^* = \frac{\alpha y}{p_1}$. Similarly $x_2^* = (1 - \alpha)y/p_2$. We call this the *(Marshallian) demand function* and it is easy to see that as income goes up so does demand for x_1 and as its price goes up demand goes down: $\partial x_1^*/\partial y = \alpha/p_1 > 0$ and $\partial x_1^*/\partial p_1 = -\alpha y/p_1^2 < 0$.

The Marshallian demand function can be contrasted with the Hicksian or *compensated demand function*. The compensated demand function asks how demand changes as prices changes when the consumer is constrained to be at the same level of utility. That is, whereas the Marshallian demand function allows utility to increase as prices drop, the Hicksian function keeps utility constant at the new prices. Consequently, the Hicksian function only has a substitution effect, whereas the Marshallian function has both substitution and demand effects.

We get there in a couple of slightly complex steps which are important nonetheless in consumer theory. We start by re-expressing the utility function $U(x_1, x_2)$ as the *indirect utility function* $V(y, p_1, p_2)$, where we calculate utility from the prices and income, assuming optimal demand. We do this by replacing x_1 and x_2 in $U(.)$ with x_1^* and x_2^* as defined above. This gives us:

$$V(y, p_1, p_2) = \left(\frac{\alpha y}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)y}{p_2}\right)^{1-\alpha} = \alpha^{\alpha}(1-\alpha)^{1-\alpha}\frac{y}{p_1^{\alpha}p_2^{1-\alpha}} = A\frac{y}{p_1^{\alpha}p_2^{1-\alpha}} \tag{10}$$

We now create an exciting new function called the *expenditure function*, which is defined as the necessary expenditure / income needed to obtain a fixed level of indirect utility $\bar{V}$. We get this basically by solving for y in the above equation.

$$e(p_1, p_2, \bar{V}) = \frac{\bar{V}p_1^{\alpha}p_2^{1-\alpha}}{A} \tag{11}$$

Now think about what $e(p_1, p_2, \bar{V})$ is - it is the analog of income y that we have seen throughout. This means that when we take the derivative of $e(.)$ with respect to p_1 this must equal the increase in the quantity of good 1 x_1 consumed. Think back to the budget constraint - a rise in p_1 leads to an increase in money spent y of x_1. But we also have an optimality condition for e in the equation above. That is, we know $\partial e/\partial p_1$ must equal $\hat{x}_1$, where we give it a hat to differentiate it from the general (uncompensated) x_1^*. And if we take the actual derivative of e we can solve for $\hat{x}_1$. So...

$$\frac{\partial e}{\partial p_1} = \frac{\alpha \bar{V} p_1^{\alpha-1} p_2^{1-\alpha}}{A} = \hat{x}_1 \tag{12}$$

Now how does this $\hat{x}_1$, which we have constrained to keep the consumer at $\bar{V}$ relate to the uncompensated x_1^* we had earlier? The best way of examining this is to compare the effects of a change in p_1 on x_1^* and on $\hat{x}_1$.

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{\alpha y}{p_1^2} \tag{13}$$

$$\frac{\partial \hat{x}_1}{\partial p_1} = \frac{(\alpha - 1)\alpha y}{p_1^2} \tag{14}$$

To see the difference between the two we subtract the latter effect from the former to get:

$$\frac{\partial x_1^*}{\partial p_1} - \frac{\partial \hat{x}_1}{\partial p_1} - = \frac{\alpha y}{p_1^2} - \frac{(\alpha - 1)\alpha y}{p_1^2} = \frac{-\alpha^2 y}{p_1^2} = I \tag{15}$$

This result shows us that the effect of a price change on uncompensated demand is larger than on compensated by an amount I, which conveniently stands for the *income effect*. Consequently, the effect of prices on the compensated demand function x_1^* is the full *substitution effect* and the effect of prices on the uncompensated demand function $\hat{x}_1$ equals the combination of the income and substitution effects.

3 General Equilibrium

The previous results focus solely on the decision of one economic agent. A *genera equilibrium* needs to consider the decisions of multiple consumers, and indeed also of firms. We begin this section following Przeworski in considering a pure exchange economy without production before conducting an (incomplete) analysis of the role of production.

3.1 Equilibrium in a Pure Exchange Economy

Przeworski uses the *Edgeworth Box* concept in analyzing an exchange economy. Here we now assume there are two consumers and two goods. We also establish a maximum level of production of our two good X_1, X_2. Consumer A consumes a share x_1 and consumer B a share $y_1 = X_1 - x_1$ of good one, and for good two we have x_2 for A and $y_2 = X_2 - x_2$ for B. Starting from some initial endowment of the goods $(x_1, y_1; x_2, y_2)$ and allowing the two consumers to constlessly trade with one another, where will we end up in equilibrium?

Well, we already know from our analysis of consumer behavior that in equilibrium, consumers will choose a package of goods such that $MRS = p_1/p_2$. Assuming both individuals are pricetakers, this implies that at equilibrium they must exchange with one another until $MRS_A = MRS_B = p_1/p_2$. Within an Edgeworth Box this means they will trade to a position where their indifference curves are tangent (that is where their MRS's equal one another) and that at this point their indifference curves will equal the price ratio. At any point that fulfils this criterion, no consumer can be made better off without making the other worse off and thus the outcome is Pareto optimal and 'efficient'.

This provides us with the *First Fundamental Theorem of Welfare Economics*, which states that given complete markets and costless exchange, consumers will trade to a position that is Pareto optimal and thus market equilibria are efficient.

It turns out, however, that $MRS_A = MRS_B$ along a whole series of points within the Edgeworth box. Or put differently, for any given indifference curve of A's, it will have a point at which it's slope equals the price-ratio as does the slope of B's curve. We call this series of points the *contract curve*. Given a different starting endowment in the Edgeworth Box, the consumers can exchange their way to a different point on the contract curve.

In fact, to be precise the *Second Fundamental Theorem of Welfare Economics* states that *any* point on the contract curve can be achieved by redistributing goods to a new starting endowment from which consumers can trade to the curve. Hence any point on the Pareto frontier is achievable given a different starting endowment and that could mean a point of relative equity or one of complete inequity - either way they would be Pareto optimal outcomes. Clearly, this theorem while powerful permits many distasteful outcomes.

3.2 Production and the Firm (ALSO ADVANCED)

I want to quickly talk about the classic theory of the firm, though Przeworski doesn't really do so - it will prove an important baseline for our analysis of monopoly theory in a few weeks. There are many similarities between the theory of production and that of consumption - indeed Steve Goldman who taught me micro used to say, "next week is theory of the firm but we're not going to do it since it's the inverse of the theory of consumption." And he's basically right at a general level but I still think it's useful to go over a few basic production functions. Again I borrow from Johansson.

We start with a firm that produces a single output - our friend x_1 (proper general equilibrium models would allow a choice of outputs as well but life's too short). We are also going to assume a Cobb Douglas utility function that uses labor and capital $x_1 = L^\alpha k^\beta$ - note that I have not defined the relationship between α and β and that's because the size of their sum dictates whether the production function has constant ($\alpha + \beta = 1$), decreasing ($\alpha + \beta < 1$), or increasing ($\alpha + \beta > 1$) returns to scale. We begin by examining the concept of an *isoquant* - that is a combination of L and K that produces the same level of output $\bar{x}_1$. As before we use the implicit function to derive:

$$MRTS = -\frac{dL}{dK}\bigg|_{x_1=\bar{x}_1} = \frac{\partial x_1/\partial K}{\partial x_1/\partial L} = \frac{\beta}{\alpha}\frac{L}{K} \tag{16}$$

Here I have something that looks a lot like our friend earlier the MRS: it is the *Marginal Rate of Technical Substitution* and it reflects how much less labor one needs for an extra unit of capital. Again the MRTS is decreasing in capital, just like the MRS was decreasing in x_1.

Now let's look at the budget constraint. We can set this as $\bar{c} = wL + rK$ where $\bar{c}$ is the total cost of production and w is the wage rate for labor and r is the interest rate on capital. We can quickly calculate, either by substitution or by the implicit function theorem that $-\frac{dL}{dK} = \frac{r}{w}$. Then we could go through the whole Lagrangian rigamarole, attempting to minimize costs(as opposed to maximizing utility) or just remember the general point about indifference curves meeting budget lines and note that at equilibrium it must be that the MRTS equals the input price ratio:

$$\frac{\beta}{\alpha}\frac{L}{K} = MRTS = -\frac{dL}{dK} = \frac{r}{w} \tag{17}$$

And this, as before with consumption, provides us with information about optimal choices of labor and capital given prevailing wage and interest rates. In fact, in a fully specified general equilibrium with two goods and two input factors, we could show that in equilibrium $MRS = p_1/p_2$, $MRTS = r/w$, $MRS = MRTS$ and $MC_1 = p_1$, where the final equality shows that marginal costs equal marginal prices. This is the full set of general equilbrium conditions in the Walrasian market. Let's move on to briefly consider this last point.

To do so we will now turn to profit-maximization (as opposed to the basic cost minimization problem). Let's assume that we are attempting to maximize short-run profits, so-called because in the short term we fix capital to equal one. Thus the firm has a fixed cost of $r \cdot 1$ and can vary its total cost by changing its labor inputs wL. And let's do one other easy thing by setting $\alpha = 1/2$ so now our production function is: $x_1 = L^{1/2} \cdot 1 = L^{1/2}$ and therefore $L = x_1^2$. We can now move to the profit function, which is revenues (prices times output) minus variable costs (labor) and fixed costs (capital):

$$\pi = p_1 x_1 - wL - r = p_1 x_1 - w x_1^2 - r \tag{18}$$

Maximizing this expression with respect to x_1 we get $p_1 = 2wx_1 = MC$, where we know that $2wx_1$ is the marginal cost of increasing x_1. This produces our last equilibrium concept. Note though that we can take this equality and use it plug in for total supply of x_1 given prices $x_1 = \frac{p_1}{2w} > 0$ which gives us the standard result that firms will increase their supply if prices go up. And secondly we can also look at their response to factor inputs. We know that $L = x_1^2 = \frac{p_1^2}{4w^2}$ and we can now find the effect of increase in wages on labor demand: $\partial L/\partial w = -\frac{2p_1^2}{w} < 0$, which is unsurprisingly negative.

4 Public Goods and Externalities

So far, as Przeworski notes, we have looked at the 'market miracle' of general equilibrium. But it turns out that this equilibrium and its relative efficiency are quite fragile to altered assumptions. I am going to leave monopolies until the week on the state but Przeworski does quickly cover increasing returns. Instead, for now I will look at public goods and externalities - both of which constitute the classic problem of market failure even in competitive markets.

4.1 Public Goods

We begin with public goods - goods with zero marginal cost for adding an extra user but with a fixed price to provide, which obviously causes problems in the standard microeconomic framework. I will borrow from Przeworski for this section.

Individual utilities come from individual consumption of private goods x_i and individual benefits from the *whole* public good G: $U_i(x_i, G)$. Let's begin by assuming individuals contribute to the public good, each with g_i, where $G = \sum_i g_i$. Then they maximize utility with respect to a budget constraint $y_i = p_x x_i + p_G g_i$. I note that since $\partial G/\partial g_i = 1$ it must be the case that $\partial U_i/\partial g_i = (\partial U_i/\partial G)(\partial G/\partial g_i) = \partial U_i/\partial G$. From our standard welfare economics analysis we know that for two goods like this the optimality condition for the individual is:

$$MRS = \frac{\partial U_i / \partial G}{\partial U_i / \partial x_i} = \frac{p_G}{p_x} \tag{19}$$

The problem is that the individual benefits from the whole of G not just g_i. Let's see what would be socially optimal. We now introduce the concept of a social planner, who is attempting to maximize the sum of utilities in societyThe social planner maximizes $\sum_i \gamma_i U_i$, where γ_i is the weight attached to each individual; subject to the budget constraint that total income pays for all the private goods and the public good: $\sum_i y_i = \sum_i p_x x_i + p_G G$. We can write this as a Lagrangian and take first order conditions:

$$L = \sum_i \gamma_i U_i + \lambda \left[\sum_i y_i - \sum_i p_x x_i - p_G G \right] \tag{20}$$

$$\sum_i \gamma_i \frac{\partial U_i}{\partial G} = \lambda p_G \tag{21}$$

$$\gamma_i \frac{\partial U_i}{\partial x_i} = \lambda p_x \quad \forall i \tag{22}$$

We now can substitute for γ_i and get:

$$\sum_i \lambda p_x \frac{\partial U_i / \partial G}{\partial U_i / \partial x_i} = \lambda p_G \tag{23}$$

And finally rearranging we get the SOCIAL OPTIMUM:

$$\sum_i \frac{\partial U_i / \partial G}{\partial U_i / \partial x_i} = \frac{p_G}{p_x} \tag{24}$$

Hmmm... this looks a little like our standard MRS for Pareto optimality -

except for the summing. Let's strip out individual i from the sum.

$$\frac{\partial U_i/\partial G}{\partial U_i/\partial x_i} + \sum_{j \neq i} \frac{\partial U_j/\partial G}{\partial U_j/\partial x_j} = \frac{p_G}{p_x} \tag{25}$$

And by rearranging we obtain:

$$\frac{\partial U_i/\partial G}{\partial U_i/\partial x_i} = \frac{p_G}{p_x} - \sum_{j \neq i} \frac{\partial U_j/\partial G}{\partial U_j/\partial x_j} \tag{26}$$

Notice that the left-hand side of this equation is now smaller than the standard MRS assumption for optimality where the MRS would equal the price ratio - it now equals something less than the price ratio. This means that society would choose *more* public goods than individuals would do on their own.

4.2 Externalities

In this final section we run through the mathematics of externalities. To mix things up a little I now follow Mueller's analysis in Chapter Two of Public Choice. Let us imagine two individuals: A and B who both consume good X but A also consumes good E for externality. Let us assume that each can choose their preferred levels of X but only A chooses E even though it also affects B (i.e. enters B's utility function). Thus $U_A = U_A(X_A, E_A)$ and $U_B = U_B(X_B, E_A)$. You can already see trouble brewing here... In standard fashion A solves the following Lagrangian:

$$L_A = U_A(X_A, E_A) + \lambda(Y_A - p_x X_A - p_E E_A) \tag{27}$$

The first order conditions for choices of X_A and E_A are as normal:

$$MRS_A = \frac{\partial U_A/\partial E}{\partial U_A/\partial X_A} = \frac{p_E}{p_x} \tag{28}$$

However, B's utility function is also affected by E_A even though B does not choose it. To solve for the social optimum we can apply the concept of Pareto-optimality - that is, we will alter the provision of goods holding B's utility *constant* at $\bar{U}_B$. To do so we use the mega-Lagrangian:

$$L_{PO} = U_A(X_A, E_A) + \lambda(\bar{U}_B - U_B(X_B, E_A)) + \gamma(Y_A + Y_B - P_x X_A - p_X X_B - p_E E_A) \tag{29}$$

And we take first order conditions w.r.t X_A, X_B, E_A.

$$\frac{\partial U_A}{\partial X_A} = \gamma P_x \tag{30}$$

$$-\lambda \frac{\partial U_B}{\partial X_B} = \gamma P_x \tag{31}$$

$$\frac{\partial U_A}{\partial E_A} - \lambda \frac{\partial U_B}{\partial E_A} = \gamma P_E \tag{32}$$

Solving for γ and λ (slightly longwinded) we get:

$$\frac{\partial U_A}{\partial E_A} + \frac{\partial U_A / \partial X_A}{\partial U_B / \partial X_B} \frac{\partial U_B}{\partial E_A} = \frac{\partial U_A}{\partial X_A} \frac{P_E}{P_x} \tag{33}$$

And then we divide by $\partial U_A / \partial X_A$ and rearrange to get:

$$MRS_{PO} = \frac{\partial U_A / \partial E_A}{\partial U_A / \partial X_A} = \frac{P_E}{P_x} - \frac{\partial U_B / \partial E_A}{\partial U_B / \partial X_B} \tag{34}$$

And here we note that the MRS under Pareto optimality is quite different from that A chooses alone, provided that $\partial U_B / \partial E_A \neq 0$. In particular if it is a positive externality then $\partial U_B / \partial E_A > 0$ and $MRS_{PO} < MRS_A$ and consequently there is undersupply of the good producing the externality if only A chooses (since $U()$ is concave). If conversely $\partial U_B / \partial E_A < 0$ then $MRS_{PO} > MRS_A$ and there is oversupply of the good by A if there is no coordination.